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Critical exponent γ for self-avoiding walks on the Sierpinski gasket family of fractals

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Abstract. We apply the Monte Carlo renormalization group (MCRG) analysis of self-avoiding walks (SAWs) on fractals to calculate the critical exponent γ, associated with the total number of distinct SAWs. In the case of the Sierpinski gasket family of fractals (whose members are labelled by an integer b, $2 \leqslant b < \infty$) we have calculated γ for $2 \leqslant b \leqslant 80$. Our MCRG results deviate at most 0.2% from the available exact results ($2 \leqslant b \leqslant 8$). The entire set of our results demonstrates that γ, being always larger than the Euclidean value $\frac{43}{32}$, monotonically increases with b.

Do critical exponents of the self-avoiding walks (SAWs) on fractals approach the Euclidean values when the underlying fractal lattices become almost Euclidean? This interesting question can be attacked in a systematic way by studying SAWs on families of fractals whose fractal (d_f) and spectral (d_s) dimensions gradually approach the corresponding Euclidean values. This kind of study was undertaken in [1] in the case of the Sierpinski gasket (SG) family of fractals, whose members can be labelled by an integer b ($2 \leqslant b \leqslant \infty$) (when $b \to \infty$, both d_f and d_s tend to their Euclidean value 2). By applying an exact renormalization group (RG) technique [1] critical exponents of SAWs were calculated for $2 \leqslant b \leqslant 8$. It turned out that the critical exponent γ (which governs, together with the connectivity constant μ, the scaling law $C_N \sim \mu^N N^{\gamma-1}$ for the total number C_N of distinct SAWs of N steps) was always larger than the Euclidean value $\gamma = \frac{43}{32}$ [2] and displayed a clear sign of monotonic increase with b. The exact RG calculation of γ beyond $b = 8$ required an unavailable computer time (roughly speaking, to get γ for $b = 9$ would take more than 85 days of continuous operating of the IBM 3090 mainframe). In this work we use the Monte Carlo renormalization group (MCRG) method, for ($2 \leqslant b \leqslant 80$), and demonstrate that γ continues to depart from $\frac{43}{32}$, that is, it continues to increase monotonically beyond $b = 8$.

In what follows we shall first explain the way we calculated γ, and then we shall present our findings, together with a discussion concerning their relevance to the current knowledge of SAWs on fractals. The MCRG method for calculating γ is a generalization of a similar method applied to calculating the critical exponent ν of SAWs for the extended sequence of the SG fractals ($2 \leqslant b \leqslant 80$) [3]. It starts by recalling the fact that each member of the SG fractal family can be constructed in stages. At the initial stage ($r = 1$) of the construction there is an equilateral triangle (generator) that contains b^2 identical smaller triangles of unit side length, out of which only the upper-oriented are physically present. The subsequent fractal stages are constructed self-similarly, so that the complete fractal is obtained in the

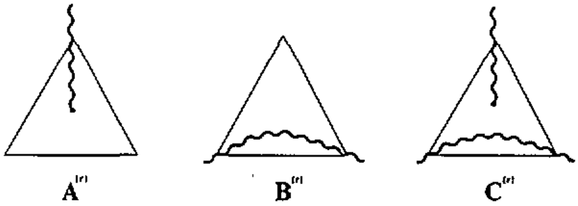

Figure 1. Schematic representation of the three restricted partition functions (for an rth-stage fractal construction) used in the calculation of the SAW critical exponent γ. The interior structure of the rth-order fractal triangle is not shown (it is manifested by the wiggles of the SAW paths).

limit $r \to \infty$. In the case of the critical exponent γ, we need the three restricted partition functions $A^{(r)}$, $B^{(r)}$, and $C^{(r)}$, that represent possible configurations of SAW within the rth stage fractal construction (see figure 1). It can be verified [1] that these functions satisfy the following recursion relations:

$$B^{(r)} = \sum_{N=b}^{b(b+1)/2} l_N (B^{(r-1)})^N \tag{1}$$

$$A^{(r)} = a_1(B^{(r-1)})A^{(r-1)} + a_2(B^{(r-1)})C^{(r-1)} \tag{2}$$

$$C^{(r)} = c_1(B^{(r-1)})A^{(r-1)} + c_2(B^{(r-1)})C^{(r-1)} \tag{3}$$

where l_N is number of all possible SAWs of N steps that traverse the fractal generator, while a_1, a_2, c_1, and c_2, are some polynomials in terms of $B^{(r-1)}$. Accepting the above relations as the RG equations, the critical exponent γ can be expressed [1] in the following way:

$$\gamma = \frac{\ln(2\lambda_2^2/b(b+1))}{\ln \lambda_1} \tag{4}$$

where λ_1 is the eigenvalue of the RG equation (1) and λ_2 is given by

$$\lambda_2 = \tfrac{1}{2}\left\{a_1(B^*) + c_2(B^*) + \sqrt{[a_1(B^*) - c_2(B^*)]^2 + 4c_1(B^*)a_2(B^*)}\right\} \tag{5}$$

with B^* being the fixed-point value of (1).

In the framework of the MCRG approach, one starts by analysing (1) for $r = 1$, and, for the sake of simplicity writes B' and B, instead of $B^{(1)}$ and $B^{(0)}$, respectively. Thus, for the one-step weight $B = 1$, B' appears to be the sum of all possible SAWs (of various lengths) that traverse the fractal generator. Furthermore, for arbitrary $B \leqslant 1$, the quantity B' can be considered as the grand-canonical partition function for the ensemble of all pertinent SAWs [3, 4]. Consequently, $\lambda_1 = (\mathrm{d}B'/\mathrm{d}B)|_{B^*}$ can be equated with the ensemble average number of steps $\langle N(B^*)\rangle$, made by all possible SAWs that traverse the fractal generator (assuming that each step of walks is weighted by B^*). For specific calculations, it is important that both quantities B^* and $\langle N(B^*)\rangle$ can be directly measured in the Monte Carlo (MC) simulation. In a quite similar way, the polynomials a_1, a_2, c_1, and c_2, can be viewed as four grand-canonical partition functions of four different SAW configurations (see figure 2), and their values at B^* can be directly measured in the MC simulations. The latter measurement can

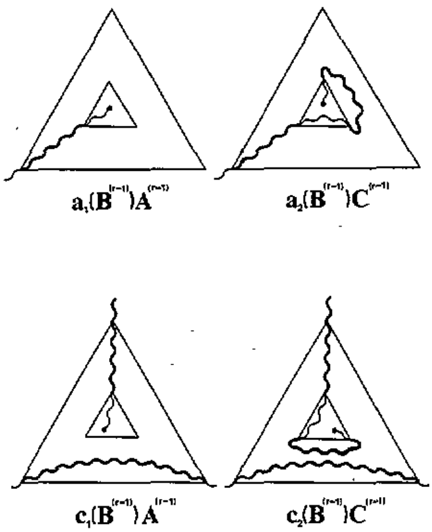

Figure 2. A diagrammatic representation of the four possible SAW configurations whose statistical weights (for the rth stage fractal construction) are given by the terms on the right-hand side of (2) (the first row of this figure) and (3) (the second row). The full wiggled segments correspond to the polynomials $a_1(B^{(r-1)})$, $a_2(B^{(r-1)})$, $c_1(B^{(r-1)})$, and $c_2(B^{(r-1)})$.

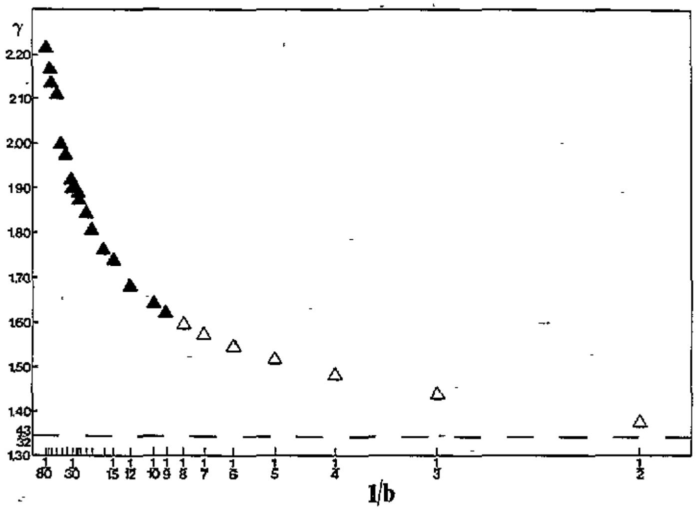

Figure 3. The exact (open triangles) and MCRG (full triangles) results for the critical exponent γ of SAWs on the SG fractals. The error bars related to the MCRG results lie within the drawn triangles. The horizontal broken line represents the Euclidean value $\gamma = \frac{43}{32}$.

be accomplished by recording all realizations of the appropriate four SAW configurations (on the fractal generator) for each MC simulation, and, finally, by dividing particular sums of the four recorded numbers by the total number of simulations. For instance, to determine $a_1(B^*)$, we let the walker start his walking (with the one-step weight B^*) at one fixed corner of the fractal generator, and record all possible walks that are terminated by entering a unit triangle (see figure 2). We repeat this MC simulation L times and finally we obtain $a_1(B^*)$ by dividing the sum of recorded numbers of walks by L. Therefore, all requisite quantities that appear in (4) and (5) can be obtained through the MC simulations, and, thereby, we can find specific values of the SAW critical exponent γ. Our results are presented in table 1 and figure 3.

Table 1. The MCRG ($2 \leqslant b \leqslant 80$) results obtained in this work for the RG eigenvalue λ_2 and the SAW critical exponent γ. For the sake of comparison, we also give the available exact RG results [1] for γ, for $2 \leqslant b \leqslant 8$. To illustrate the amount of the computer work needed for determination of λ_2, we quote here that for $b = 80$; for instance, it was necessary to run a PC with the Intel 80486 processor for 170 hours. Finally, the table is completed by quoting values of B^* and λ_1 [3] that were used in (4) and (5) to calculate γ.

b	Number of MC realization	B^*	λ_1	λ_2	γ
2	exact				1.3752
	5×10^5	0.61825 ± 0.00061	2.382 ± 0.001	3.146 ± 0.002	1.3750 ± 0.0026
3	exact				1.4407
	5×10^5	0.55137 ± 0.00044	3.992 ± 0.003	6.641 ± 0.008	1.4410 ± 0.0024
4	exact				1.4832
	5×10^5	0.50658 ± 0.00034	5.805 ± 0.004	11.67 ± 0.02	1.4852 ± 0.0024
5	exact				1.5171
	5×10^5	0.47455 ± 0.00028	7.790 ± 0.006	18.40 ± 0.04	1.5184 ± 0.0025
6	exact				1.5467
	5×10^5	0.45091 ± 0.00024	9.942 ± 0.008	27.18 ± 0.06	1.5500 ± 0.0026
7	exact				1.5738
	5×10^5	0.43240 ± 0.00021	12.23 ± 0.01	38.0 ± 0.1	1.5759 ± 0.0027
8	exact				1.5991
	5×10^5	0.41780 ± 0.00019	14.67 ± 0.01	51.5 ± 0.2	1.6015 ± 0.0028
9	5×10^5	0.40574 ± 0.00017	17.21 ± 0.01	67.3 ± 0.2	1.6202 ± 0.0030
10	5×10^5	0.39586 ± 0.00007	19.908 ± 0.008	86.9 ± 0.3	1.6454 ± 0.0028
12	5×10^5	0.38037 ± 0.00013	25.64 ± 0.02	135.4 ± 0.6	1.6827 ± 0.0033
15	5×10^5	0.36396 ± 0.00011	34.95 ± 0.03	239 ± 1	1.7347 ± 0.0037
17	5×10^5	0.35593 ± 0.00008	41.79 ± 0.03	333 ± 2	1.7641 ± 0.0038
20	5×10^5	0.34681 ± 0.00006	52.59 ± 0.03	523 ± 4	1.8096 ± 0.0041
22	5×10^5	0.34197 ± 0.00008	60.31 ± 0.05	702 ± 6	1.8473 ± 0.0045
25	5×10^5	0.33602 ± 0.00008	72.44 ± 0.06	1000 ± 10	1.8762 ± 0.0048
26	5×10^5	0.33444 ± 0.00007	76.61 ± 0.07	1130 ± 10	1.8908 ± 0.0050
27	5×10^5	0.33285 ± 0.00006	81.04 ± 0.06	1260 ± 10	1.8973 ± 0.0050
30	5×10^5	0.32876 ± 0.00007	94.33 ± 0.08	1700 ± 20	1.9207 ± 0.0055
35	5×10^5	0.32350 ± 0.00008	117.6 ± 0.1	2830 ± 40	1.9816 ± 0.0061
40	5×10^5	0.31936 ± 0.00006	142.9 ± 0.1	4120 ± 60	2.0024 ± 0.0068
50	5×10^5	0.31396 ± 0.00007	197.8 ± 0.1	9500 ± 200	2.1133 ± 0.0077
60	5×10^5	0.31011 ± 0.00006	256.7 ± 0.2	16100 ± 400	2.1380 ± 0.0086
70	5×10^5	0.30745 ± 0.00006	322.1 ± 0.6	25900 ± 700	2.1658 ± 0.0101
80	5×10^5	0.30546 ± 0.00006	391.6 ± 0.7	43000 ± 1000	2.2177 ± 0.0119

The data given in table 1 and figure 3 reveal several interesting facts. First, one can notice from table 1 that the MCRG results, obtained in this work, deviate at most 0.2% from the exact RG results found [1] for $2 \leqslant b \leqslant 8$. Next, we observe that all values of γ, from the entire set $2 \leqslant b \leqslant 80$, are larger than the Euclidean value $\frac{43}{32}$ predicted [2] for two-dimensional regular lattices. In addition, it appears that the available values of γ display a monotonic increase with b, which leads one to assume that γ will continue to increase beyond $b = 80$. This assumption is in accord with the finite-size scaling argument [5] which predicts that γ, for the SG fractals, should converge to the non-Euclidean value $\frac{133}{32}$ from below, when $b \to \infty$. Thus our results offer a support to the finite-size scaling argument (although the range of results is still not sufficiently large to allow a numerical test of the asymptotic behavior of γ).

The established behaviour of γ for the SG fractals is in accord with the results obtained for the plane-filling (PF) family of fractals, which also displayed a monotonic increase (above the Euclidean value) for the fractal enumerator b being between 3 and 121 [6]. However, both groups of results (for the SG and PF fractals) are in disagreement with the various arguments [7–10] which state that γ for SAWs on the critical percolation clusters is not different from γ of SAWs on fully occupied Euclidean lattices. At this point, it may be argued that the SG and PF fractals are not archetypes of the percolation clusters, and that the observed disagreement may stem from the basic difference between deterministic and random fractals. Yet, in view of the fact that the critical exponents of SAWs on the percolation clusters still comprise a controversial research problem, the noted disagreement with the exact and MCRG results calls for additional studies.

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